Jérôme, Algorithmic Game Theory and Strategyproof Facility Location

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Based on joint work with Laurent Gourvès (LAMSADE), Jérôme Monnot (LAMSADE) and Panagiotis Patsilinakos (NTUA)

Homage to Jérôme Monnot, Univ. Paris-Dauphine, December 2021
Spanning Trees Constructed by Selfish Agents
[Gourvés M., WINE 08]

- Agents pay for **first edge** of path to root.
- Pure Nash equilibrium iff **strong** PNE.
- $\text{PoA} = \min\{\Theta(\log n), \text{depth of MST} \}$
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Strong Equilibria of (k-)Cut Games [Gourvés, M., WINE 09, TAMC 10]
- Cut games admit strong PNE and SPoA = 2/3.
- k-Cut games admit 3-strong PNE.
- If k-Cut games admit strong PNE (conjectured), SPoA = \( \frac{2k-2}{2k-1} \).

Congestion Games with Capacited Resources
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PNE exists, m = 2 resources, NP-hard to tell for m ≥ 3.
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- **Bus route** selection, based on flow time from entering bus until destination.
- PNE exist (and computed **efficiently**), if buses have “identical routes”, or $m = 2$ and **metric** distances, or distances $\in \{1, 2\}$.
- PoA $\leq n$ for metric distances.
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**And Much More ...**

- **Coordination mechanisms** for scheduling selfish tasks with setup times [Gourvés, M., Telelis, WINE 09]
- Selfish graph **coloring** [Escoffier, Gourvés, M., CIAC 10]
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- Fair allocation of indivisible goods: approximation of the **maximin share** for $n = 3$ and matroids [Gourvés, M., CIAC 17]
Our Focus: $k$-Facility Location Games

Public Good Allocation for Strategic Agents on the Line

- Agents $N = \{1, \ldots, n\}$ on the real line.
- Agent $i$ wants a facility close to $x_i$, which is private information.
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(Randomized) Mechanism

Mechanism \(F\) maps reported ideal locations \(y = (y_1, \ldots, y_n)\) to (probability distribution over) set(s) of \(k\) facilities.
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Public Good Allocation for Strategic Agents on the Line

- Agents $N = \{1, \ldots, n\}$ on the real line.
- Agent $i$ \textbf{wants} a facility close to $x_i$, which is \textbf{private information}.
- Each agent $i$ \textbf{reports} $y_i$ that may be \textbf{different} from $x_i$.

(Randomized) Mechanism

\textbf{Mechanism} $F$ maps reported ideal locations $y = (y_1, \ldots, y_n)$ to (probability distribution over) set(s) of $k$ \textbf{facilities}.
Connection Cost

(Expected) distance of agent $i$’s **true location** to the **nearest** facility:

$$
\text{cost}[x_i, F(y)] = \text{dist}(x_i, F(y)) = \min_{c \in F(y)} |x_i - c|
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Preferences and Truthfulness

Connection Cost

(Expected) distance of agent $i$’s true location to the nearest facility:

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Truthfulness

For any location profile $x$, agent $i$, and location $y$:

$$\text{cost}[x_i, F(x)] \leq \text{cost}[x_i, F(y, x_{-i})]$$
Candidate Facility Locations:

- **Unrestricted**: Any point (esp. agent locations) can be facility.
- **Restricted**: Facilities selected from $m$ candidate locations $C$.

Motivation from multi-winner elections: Chamberlin-Courant.
Variants and Social Efficiency

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Social Objective

$F(x)$ should optimize (or approximate) a given **objective function**.

- **Social Cost**: minimize $\sum_{i=1}^{n} \text{cost}[x_i, F(x)]$
- **Social Welfare**: maximize $\sum_{i=1}^{n} (L - \text{cost}[x_i, F(x)])$
Median Mechanism

- Median of \((x_1, \ldots, x_n)\): truthful and optimal, when unrestricted.
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- Candidate location closest to \(\text{med}(x_1, \ldots, x_n)\): truthful and **1/3-approximate**, when restricted.
1-Facility Location on the Line

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- Anonymity and truthfulness iff **generalized** median [Moulin 80]
Percentile Mechanisms [Sui Boutilier Sandholm, IJCAI 13]

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\((\alpha_1, \ldots, \alpha_k)\)-percentile mechanism \((0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq 1)\):

- vote\((\ell)\) = \#agents preferring \(\ell \in C\) to other candidates in \(C\).
- \(j\)-th facility at leftmost \(\ell \in C\) with \(\geq \alpha_j\) fraction of vote on \(\ell\) and its left.
  - Median is 0.5-percentile. Two-Extremes is \((0, 1)\)-percentile.

\(n = 80\) \hspace{1cm} \(k = 4\) \hspace{1cm} \((0.1, 0.3, 0.5, 0.9)\)-percentile
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  - Median is 0.5-percentile. Two-Extremes is \((0, 1)\)-percentile.
  - Percentile mechanisms are anonymous and truthful.

\[ n = 80 \quad k = 4 \quad (0.1, 0.3, 0.5, 0.9)\)-percentile
Robust version of \((kq)\)-Facility Location where each agent receives utility from \(q\) different facilities.

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- Single-peaked preferences: in any set of \(kq\) facilities, each agent receives utility from \(q\) consecutive facilities.
- Generalization of single-peakedness and percentile mechanisms to tuples of \(q\) consecutive candidate locations.
- For any \(q \geq 1\), \((1/(2k), 3/(2k), \ldots, (2k − 1)/(2k))\)-percentile applied to \(q\)-tuples is truthful and \((2k − 3)/(2k − 1)\)-approximate.
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- Generalization of single-peakedness and percentile mechanisms to \(q\) consecutive candidate locations.
- For any \(q \geq 1\), \(\frac{1}{(2k)}, \frac{3}{(2k)}, \ldots, \frac{2k - 1}{(2k)}\)-percentile applied to \(q\)-tuples is \textbf{truthful} and \(\frac{2k - 3}{(2k - 1)}\)-approximate.
- Optimal solution through generalization of LP-based approach in [Hajiaghayi et al., SODA 14]
Truthful Location of 2 Facilities

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- Bounded approximation requires facility in **each optimal** cluster. But optimal clustering is **sensitive** to agent deviations.
- Focus on instances with **stable** optimal clustering.
Perturbation Stability for $k$-Facility Location

Perturbation Stability in Clustering [Bilu Linial, ITCS 10]

- $\gamma$-stability: scaling down any distances by factor $\leq \gamma$ (while maintaining metric property) does not affect optimal solution.

For $\gamma \geq 2$, (metric) $k$-Facility Location solvable in poly-time!

Angelidakis Makarychev Makarychev, STOC 17

$k$-Facility Location remains hard for $\gamma \leq 2 - \epsilon$.

Real-world instances are stable: "Clustering is hard when it doesn't matter" [Roughgarden 17]
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![Diagram](image-url)
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  - $k$-Facility Location remains **hard** for $\gamma \leq 2 - \varepsilon$.
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Question

Assume that “true” instances are indeed stable.
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Some Negative Observations

- Optimal solution not truthful for any stability $\gamma \geq 1$. 

**Truthful $k$-Facility Location in Stable Instances**

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Question

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Some Negative Observations

- Optimal solution not truthful for any stability $\gamma \geq 1$.
- For $k \geq 3$, deterministic anonymous truthful mechanisms for $(\sqrt{2} - \varepsilon)$-stable instances have unbounded approximation (based on [F. Tzamos, ICALP 13])
Truthful $k$-Facility Location in Stable Instances

Remedy and Main Results

- **Optimal** clustering $(C_1, \ldots, C_k)$ due to bounded approximation.
- Stability verification (necessary cond.): allocate facilities only if
  \[
  \max \{ \text{diam}(C_i), \text{diam}(C_{i+1}) \} < d(C_i, C_{i+1})
  \]

For $(\sqrt{2} + 3)$-stable instances without singleton clusters, optimal solution is truthful.

For 5-stable instances, facility at second from the right in each optimal cluster is truthful and $(n-2)/2$-approximate.

For 5-stable instances, facility at random agent in each optimal cluster is truthful and 2-approximate.
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- For 5-stable instances, facility at **random** agent in each optimal cluster is **truthful** and 2-approximate.
Optimal Mechanism and Approach to Truthfulness

If optimal clustering \((C_1, \ldots, C_k)\) has \textbf{singleton} clusters or \(\max\{\text{diam}(C_i), \text{diam}(C_{i+1})\} \geq d(C_i, C_{i+1})\), do not allocate facilities!

Otherwise, facilities at \((\text{med}(C_1), \ldots, \text{med}(C_k))\).
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- Key deviation: rightmost agent of \(C_i\) deviates to \(C_j\), causing \(C_j\) to split and \(C_i\) to merge with \(C_{i+1}\).  

- “Simulate” increase in cost of \(C_j\) by \(\gamma\)-perturbation and decrease in cost of \(C_j\) by agent’s cost improvement.
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- “Simulate” increase in cost of \(C_j\) by \(\gamma\)-perturbation and decrease in cost of \(C_j\) by agent’s \textit{cost improvement}.
- Stability: optimal clustering \textbf{not affected} by deviation.
Open Questions

- Close the **gap in stability** for bounded approximation: lower bound of $\sqrt{2}$ and upper bound of $2 + \sqrt{3}$ (or 5).
- Extension to **trees** and study of **general metrics**.
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- Close the **gap in stability** for bounded approximation: lower bound of $\sqrt{2}$ and upper bound of $2 + \sqrt{3}$ (or 5).
- Extension to **trees** and study of **general metrics**.
- Possibility of truthfulness **for all** instances and bounded approximation only **for stable**? (conjecture: **no**)
- Complexity of **determining** whether a $k$-Facility Location instance is **$\gamma$-stable**, esp. for line and trees?
Thank You for Everything and Goodbye

We, your many friends, deeply miss your kindness, openness, collaboration, passion for research and life, warm smile and true love for people, and so many things we kept learning from you.

Thank you and goodbye, Jérôme